An Explicit Derivation of the Relationships Between the Parameters of an Interdigital Structure and the Equivalent Transmission-Line Cascade

HENRY J. RIBLET, FELLOW, IEEE

Abstract—The general *n*th-order admittance matrix for an array of parallel conductors placed between ground planes is exhibited, subject to the assumption that direct coupling exists only between adjacent conductors and that only a TEM wave is present whenever all the conductors but one are grounded.

When the alternate terminals of one of these arrays are connected to ground, as in an interdigital bandpass filter, the admittance matrix yields a subsystem of equations which, except for sign, is identical in form with the node equations of suitably selected transmission-line cascade. The identification of the coefficients in these similar systems of equations explicitly determines the coefficients of the admittance matrix in terms of the parameters of the prototype transmission-line cascade. In turn, the capacities to ground and the mutual capacities, each per unit length, for the array of parallel conductors can be determined from the coefficients of its admittance matrix by imposing the pertinent voltage conditions on the admittance equations. Thus, one arrives, explicitly, at the general formulas used in the design of interdigital filters which relate the capacities per unit length of the parallel conductors to the parameters of the prototype transmission-line cascade.

It is shown that, if the first element of the interdigital structure is open-circuited, the transmission-line cascade begins with a series, open-circuited quarter-wave stub while, if the first element of the interdigital structure is short-circuited, the first element of the cascade is a shunt, short-circuited quarter-wave stub. Extensions of the method to equivalences with other prototype networks are suggested.

In the Appendix, closed expressions for the self and mutual admittances of the parallel conductor array are given in terms of the self impedances and coupling coefficients of the *n*th-order impedance matrix proposed by Bolljahn and Matthaei for this structure, subject to the assumption of no coupling between nonadjacent conductors. These are shown to be consistent with the requirement that the admittance matrix be the reciprocal of the impedance matrix.

Introduction

DESIGN EQUATIONS for interdigital bandpass filters which assume an exact equivalence between these structures and a transmission-line cascade comprised of alternate equal-length line sections and shunt or series stubs of the same length have been given by Matthaei [1]. He justified this equivalence by applying a "folding operation" to the dual of the parallel-coupled filter analyzed by Cohn [2]. Recently, Wenzel [3] has inferred that this equivalence is a rigorous consequence of the impedance matrix, assumed by Bolljahn and Matthaei [4] in their discussion of the general properties of parallel conductors between ground planes for the case when there is no coupling between nonadjacent conductors. This inference was based on the important discovery[1] by Wenzel that the second-order impedance matrix of the interdigital filter is equivalent to the second-order impedance matrix of a suitable transmission-line cascade, for the special cases of symmetric networks with up to eight lines and for asymmetric networks with up to four lines.

This paper proceeds from the admittance equations for the array of parallel conductors rather than from the impedance equations. This seemingly minor change in the point of view, however, results in major simplification of the analysis. In fact, the required form of the admittance equations can be deduced from two electrical assumptions regarding the array of parallel conductors between ground planes which, in turn, are consequences of the geometry of the array. When the voltage conditions associated with an interdigital filter are imposed on the admittance equations of the parallel conductors a subsystem of equations results which can be identified term by term with the node equations of an equivalent, prototype transmission-line cascade. This correspondence is essentially the same as that obtained by Mattaei using a "folding operation." The procedure of this paper has the important advantage, however, in that it provides an explicit derivation of the relationships between the parameters of the equivalent transmission-line cascade and the capacities per unit length of the parallel conductor array.

In the Appendix, it is shown, in general, that the impedance matrix of Bolljahn and Mattaei, subject to the condition of no coupling between nonadjacent conductors, is the reciprocal of the admittance matrix used in this paper and closed expressions for the coefficients in the admittance equations in terms of the coefficients of the impedance equations are presented.

The Admittance Equations

The analysis of interdigital structures, to be given in this paper, depends on the particular form of the general admittance equations of the array of parallel conductors between ground planes. With the terminal voltages and currents defined as in Fig. 1, these may be written,

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The author is with Microwave Development Laboratories, Inc., Needham, Mass.

[1] The author wishes to acknowledge his indebtedness to this result since it was the starting point for this paper which, he believes, is a rigorous justification of Matthaei's "folding operation."

$$\begin{vmatrix} I_{1a} \\ I_{1b} \\ I_{2a} \\ I_{2b} \\ I_{3a} \\ I_{3b} \\ \vdots \\ I_{na} \\ I_{nb} \end{vmatrix} = p \begin{vmatrix} Y_{11} & -Y_{11}t & Y_{12} & -Y_{12}t & 0 & \cdot & \cdot & & & & & & \\ -Y_{11}t & Y_{11} & -Y_{12}t & Y_{12} & 0 & \cdot & \cdot & & & & & & \\ Y_{12} & -Y_{12}t & Y_{22} & -Y_{22}t & Y_{23} & -Y_{23}t & 0 & \cdot & \cdot & & & & \\ -Y_{12}t & Y_{12} & -Y_{22}t & Y_{22} & -Y_{23}t & Y_{23}t & 0 & \cdot & \cdot & & & & \\ 0 & 0 & Y_{23} & -Y_{23}t & Y_{33} & -Y_{33}t & Y_{34} & -Y_{34}t & 0 & \cdot & \cdot & & \\ \cdot & \cdot & -Y_{23}t & Y_{23} & -Y_{33}t & Y_{33} & -Y_{34}t & Y_{34} & 0 & \cdot & \cdot & & \\ & & & & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Y_{nn} & \cdot & -Y_{nn}t & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -Y_{nn}t & \cdot & Y_{nn} & & \end{vmatrix} \cdot \begin{vmatrix} V_{1a} \\ V_{1b} \\ V_{2a} \\ V_{2b} \\ V_{3a} \\ V_{3b} \\ \vdots \\ V_{na} \\ V_{nb} \end{vmatrix}, \qquad (1)$$

where $p=-j\cos\beta l/\sin\beta l$ and $t=\sec\beta l$. Moreover, Y_{ii} is a positive real number while $Y_{i,i+1}$ is a negative real number.

How these equations may be obtained by inverting the impedance matrix which Bolljahn and Matthaei employed in their discussion of interdigital structures is demonstrated in the Appendix. A simpler and possibly more useful procedure is to infer the form of the admittance equations (1) directly from the geometry of the parallel conductors of Fig. 1.

It will be useful to state these conditions on parallel conductors with terminals on the forward or "a" side as seen in Fig. 1 but of infinite extent on the far side. Both conditions will be imposed on the electromagnetic fields which result when the ith input terminal ia is subjected to a nonzero input voltage V_{ia} while all other input voltages are zero. Then,

A) The electromagnetic field extending beyond the $i-1$st and $i+1$st conductors is negligible.
B) The electromagnetic field associated with the ith conductor is entirely a TEM wave propagating in the axial direction of the conductors whose propagation constant is independent of i.

As an immediate consequence of B), the field components of any waves excited on the parallel conductors of Fig. 1, when all of the inputs are shorted except for those of the ith conductor, will be transverse to the axis of the conductors. Also, the electric and magnetic vectors are solutions of Laplace's equation and may be derived from a complex potential function. In short, we are dealing with a coaxial transmission-line in which the ith conductor is the inner conductor and the outer conductor comprises the ground planes and the other conductors, all at ground potential. If we now define Y_{ii} as the ratio of the current in the ith conductor to the voltage to ground and Y_{ij} as the ratio of the current in the jth conductor to the voltage between the ith conductor and ground, we have the following conclusions:

a) As the characteristic admittance of a coaxial line, Y_{ii} is a positive real number.
b) As the measure of a portion of the current in the *outer* conductor of a coaxial line, Y_{ij} is a negative real number.
c) $|Y_{ij}|<Y_{ii}$.

Moreover, because of condition A), $Y_{ij}=0$ if $j>i+1$.

If the infinite system of parallel conductors is terminated now at "b," equidistant on each conductor by a length l from the input at "a," standing waves will, in general, be established on each of the i transmission lines defined by the termination conditions assumed above. On each of these lines, the currents at the "i" terminals are determined from the voltages at the "i" terminals by means of the equations,

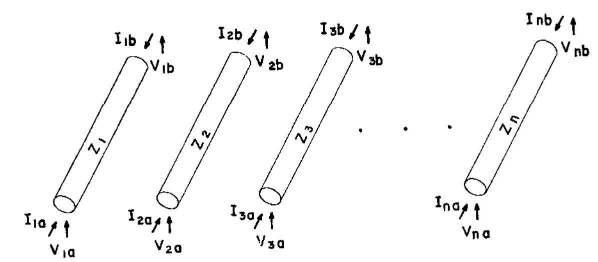

Fig. 1. Schematic array of parallel conductors between ground planes.

$$I_{ib} = pY_{ii}V_{ia} - ptY_{ii}V_{ib}$$
$$I_{ib} = -ptY_{ii}V_{ia} + pY_{ii}V_{ib}.$$

Even though the voltages at the other terminals are zero, V_{ia} and V_{ib} establish nonzero currents at the adjacent terminals. These currents, as we have seen, however, differ only from I_{ia} and I_{ib} in sign and absolute value. To obtain them, we replace Y_{ii} by $Y_{i,i+1}$ or $Y_{i-1,i}$, above. We can now reconstruct the columns of (1) from conditions A) and B). The reciprocity theorem relates terms in different columns of (1) and assures its symmetry about its principal diagonal.

How well condition A) is satisfied by a given array of parallel conductors will depend on the relative spacing of all of its components. In a general way, the coupling between nonadjacent conductors will decrease as the spacing between the conductors and the ground planes decreases and the spacing between the conductors increases. Then, in this case, condition B) places a limit only on the spacing between adjacent conducting elements.

An essential point in the design procedure for interdigital filters is the fact that the proportionality factors are identical which relate Y_{ii} and Y_{ij} to the corresponding self and mutual capacities of the two-dimensional several conductors problem with the same cross section. Consider Y_{ii} and $Y_{i,i+1}$. Condition B) assures us that the electric and magnetic fields associated with the two-conductor problem, consisting of ith conductor at one potential and the ground planes and the other conductors at zero potential can be obtained from a complex potential, $\phi+j\psi$. Thus, the axial current in any portion of ith conductor or $i+1$th conductor is given by $\sqrt{\mu/\epsilon}\,(\phi_2-\phi_1)$ where ϕ_2 and ϕ_1 are the values of the stream function ϕ at the limits of that portion of the conductor being considered. Now if the same complex potential function is used to solve the static potential problem, the total change Q, per unit length, on any portion of these conductors is given by the $\epsilon(\psi_2-\psi_1)$. Thus, we have, in general, that the current of a TEM wave on any portion of the conductors is related to the charge per unit length for the static potential problem, over the same portion of the conductor, by the relationship, $I=Q/\sqrt{\mu\epsilon}$. If we divide both I and Q by the potential difference between the conductors, we have $c_{ii}=\sqrt{\mu\epsilon}\,Y_{ii}$ and $c_{ij}=\sqrt{\mu\epsilon}\,Y_{ij}$, where c_{ii} and c_{ij} are the self and mutual capacities per unit length in the corresponding several conductor potential problem.[2]

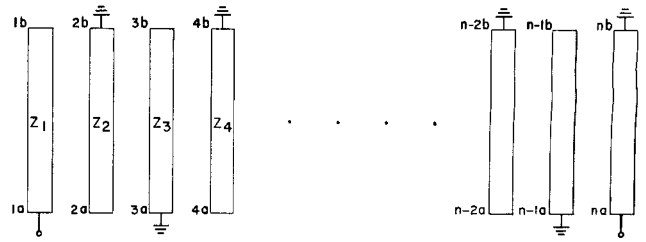

Fig. 2. Schematic interdigital structure showing terminal numbering.

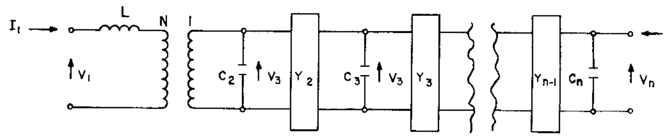

Fig. 3. Equivalent transmission-line cascade.

The Node Equations of the Transmission-Line Cascade

When we impose the terminal conditions of the interdigital structure in Fig. 2 on the admittance equations (1) of the array of parallel conductors of Fig. 1, we obtain a series of equations which can be compared term by term with the equations for the node currents in the transmission-line cascade of Fig. 3. In the derivation of these node equations, we will require the admittance equations for the series inductor, ideal transformer, transmission-line element, and shunt capacitor of Fig. 3. For the series inductor,

$$I_{\text{in}} = \frac{1}{Lp} V_{\text{in}} - \frac{1}{Lp} V_{\text{out}}$$
$$I_{\text{out}} = -\frac{1}{Lp} V_{\text{in}} + \frac{1}{Lp} V_{\text{out}}. \tag{2}$$

Here positive voltages are upward, positive currents flow toward the network and the input terminal is assumed to be at the left. For the ideal transformer,

$$V_{\text{in}} = N\,V_{\text{out}}$$
$$I_{\text{in}} = -\frac{1}{N} I_{\text{out}}. \tag{3}$$

For the transmission-line element,

$$I_{\text{in}} = YpV_{\text{in}} - YtpV_{\text{out}}$$
$$I_{\text{out}} = -YtpV_{\text{in}} + YpV_{\text{out}}, \tag{4}$$

where Y is the characteristic admittance of the transmission-line element. Finally, for a shunt capacitor, the current I flowing downward due to a voltage V impressed across its terminals is given by,

$$I = CpV. \tag{5}$$

We are now in a position to apply the Kirchhoff current law to the nodes of the circuit of Fig. 3. These are represented by hollow circles in the figure and are numbered from left to right. These nodes occur in pairs, and we shall only consider the upper nodes of each pair since the currents in the lower nodes differ only in sign from the currents in the upper nodes. A series of voltages running upward between related node pairs is shown in Fig. 3. We agree that currents flowing out of the upper node of each pair is assumed positive and notice that, except for the first and last nodes, the net current into each node is zero. In view of (2), (3), (4), and (5), we have,

$$\begin{aligned}
I_1 &= \frac{1}{Lp}V_1 - \frac{N}{Lp}V_2\\
0 &= -\frac{N}{Lp}V_1 + \left(\frac{N^2}{Lp} + C_2p + Y_2p\right)V_2 - Y_2tpV_3\\
0 &= -Y_2tpV_2 + (Y_2 + C_3 + Y_3)pV_3 - Y_3tpV_4\\
&\;\;\vdots\\
0 &= -Y_{i-1}tpV_{i-1} + (Y_{i-1} + C_i + Y_i)pV_i - Y_itpV_{i-1}\\
&\;\;\vdots\\
I_n &= -Y_{n-1}tpV_{n-1} + (Y_{n-1} + C_n)pV_n.
\end{aligned} \tag{6}$$

[2] For the definition of the terms involved in and a discussion of the several conductor problem, the reader is referred to S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio*. New York: Wiley, pp. 262–265, 1953.

It should be observed that this system of equations is similar in form to the node equations which Guillemin [5] has employed in his discussion of equivalent ladder networks in that all nonzero terms fall on the three diagonals centered about the principal diagonal. This is to be expected since there is no direct coupling between nonadjecent nodes for either form of network cascade. On the other hand, (6) is readily distinguishable from the node equations of a ladder network of lumped constant elements in the occurrence of the nonrealizable frequency variable tp in some of the off-diagonal terms.

The Problem

The principal result of this paper is the explicit proof that the interdigital structure of Fig. 2 and the transmission-line cascade of Fig. 3 have the same overall performance as seen from the input and output terminals. Matthaei has analyzed this problem in two steps. First, he has constructed a parallel-coupled filter which must have the same overall performance as the transmission-line cascade because it is a cascade of elements each of which has the same overall performance as the elements making up the transmission-line cascade; secondly, he has introduced a "folding operation" to show that the interdigital structure and the parallel-coupled filter must have approximately the same overall characteristics. Wenzel has based the characteristics of the interdigital structure on the impedance matrix of the array of parallel-conductors array exactly and shown, in a limited number of cases, that the two-port impedance matrices, determined by Z_{11}, Z_{12}, and Z_{22}, of the networks of Fig. 2 and Fig. 3 are the same.

This paper combines features of both points of view. Like Wenzel, the characteristics of the interdigital structure are determined by the immittance equations of the array of parallel conductors, but like Matthaei, the final equivalence is established on an element by element basis rather than on the overall performance of the networks. The result is a rigorous equivalence which determines the relationships between the parameters of an array of parallel conductors between ground planes and those of a transmission-line cascade, in general.

The Solution

Let us consider the interdigital filter of Fig. 2, in which the first element is open-circuited while the last element is short-circuited. Then the terminal voltage conditions[3] are,

$$V_{2b} = V_{3a} = V_{4b} = \cdots = V_{nb} = 0,$$

while the terminal current conditions are,

$$I_{1b} = I_{2a} = I_{3b} = I_{4a} = \cdots = I_{(n-1)b} = 0.$$

When these conditions are imposed on the general admittance equations (1) the following subsystem of equations results:

$$\begin{aligned}
I_{1a} &= Y_{11}pV_{1a} - Y_{11}ptV_{1b} + Y_{12}pV_{2a}\\
0 &= -Y_{11}ptV_{1a} + Y_{11}pV_{1b} - Y_{12}ptV_{2a}\\
0 &= Y_{12}pV_{1a} - Y_{12}ptV_{1b} + Y_{22}pV_{2a} - Y_{23}ptV_{3b}\\
0 &= -Y_{23}ptV_{2a} + Y_{33}pV_{3b} - Y_{34}ptV_{4a}\\
&\vdots\\
0 &= -Y_{n-2,n-1}ptV_{(n-2)a} + Y_{n-1,n-1}pV_{(n-1)b} - Y_{n-1,n}ptV_{na}\\
I_{na} &= -Y_{n-1,n}ptV_{(n-1)b} + Y_{nn}pV_{na}.
\end{aligned} \tag{7}$$

Here, except in the upper left-hand corner, all terms are zero except those that fall on the principal diagonal where they have the form $Y_{ii}p$, and on the two adjacent diagonals where they have the form $-Y_{i,i+1}pt$. This exception can be avoided by eliminating the node at $1b$ from the system. To do this multiply the second equation by t and add to the first equation and then multiply the second equation by $Y_{12}/Y_{11}t$ and add to the third equation. Now, if use is made of the fact $p(1-t^2)=1/p$, a subsystem of (7) results in which V_{1b} does not occur, having the form,

$$\begin{aligned}
I_{1a} &= Y_{11}/pV_{1a} + Y_{12}/pV_{2a}\\
0 &= Y_{12}/pV_{1a} + (Y_{12}^2/Y_{11}p + (Y_{22} - Y_{12}^2/Y_{11})p)V_{2a} - Y_{23}ptV_{3b}\\
0 &= -Y_{23}ptV_{2a} + Y_{33}pV_{3b} - Y_{34}ptV_{4a}\\
&\vdots\\
0 &= -Y_{n-2,n-1}ptV_{(n-2)a} + Y_{n-1,n-1}pV_{(n-1)b} - Y_{n-1,n}ptV_{na}\\
I_{na} &= -Y_{n-1,n}ptV_{(n-1)b} + Y_{nn}pV_{na}.
\end{aligned} \tag{8}$$

Except for certain minus signs occurring in the off-diagonal term of (6), this system has exactly the form of (6). Since these minus signs may be introduced by phase-reversing ideal transformers, we have shown a rigorous equivalence between the interdigital filter of Fig. 2 and the transmission

[3] In selecting $V_{nb}=0$, an even number of conductors has been assumed. Selecting an odd number of conductors alters the form but not the conclusions of the following arguments.

line of cascade of Fig. 3, except for phase. This equivalence permits the following identification of parameters, proceeding from left to right in the figures,

$$\begin{aligned} Y_{11} &= 1/L \\ Y_{12} &= -N/L \\ Y_{22} &= C_2 + Y_2 + N^2/L \\ Y_{23} &= -Y_2 \\ Y_{33} &= Y_2 + C_3 + Y_3 \\ &\vdots \\ Y_{n-1,n} &= -Y_{n-1} \\ Y_{n,n} &= Y_{n-1} + C_n. \end{aligned} \tag{9}$$

Here minus signs have been introduced in the equations for each of the mutual admittances $Y_{i,i+1}$, except Y_{12}. These are required since $Y_{i,j}$ are known to be negative real numbers. These minus signs imply phase reversing ideal transformers in Fig. 3, which have been omitted for the sake of convenience. Now the self and mutual capacities per unit length of the array of parallel conductors are obtained directly from (9) by use of the proportionality factor $\sqrt{\mu\epsilon}$, as we have seen.

The determination of the physical dimensions of a particular interdigital structure from known values of L, N, $\cdots$, C_n depend on calculations made by Getsinger [6] for rectangular conductors and by Cristal [7] for round rods. These calculations, in the form of graphs, express the capacity to ground of the individual conductors as well as their mutual capacity, both per unit length, normalized to the permittivity of the medium, in terms of the physical dimensions of the structure. The capacity to ground, per unit length, of the ith conductor will be denoted by c_i^g while the negative of the mutual capacity per unit length between the ith and jth conductor will be denoted by $c_{i,j}^m$.[4] Our immediate objective then is a set of equations similar to (9) expressing c_i^g and $c_{i,i+1}^m$ in terms of the parameters of the prototype transmission-line cascade.

Corresponding to c_i^g, one can define a characteristic admittance to ground of the ith conductor Y_{ii}^g as the ratio of the current flowing on the ith conductor to the voltage to ground when all the input voltages are the same and the conductors extend to infinity in the direction of the "b" terminals. Then, in view of (1) and earlier remarks,

$$\begin{aligned} Y_{11}^g &= Y_{11} + Y_{12} \\ Y_{22}^g &= Y_{12} + Y_{22} + Y_{23} \\ Y_{33}^g &= Y_{23} + Y_{33} + Y_{34} \\ &\vdots \\ Y_{nn}^g &= Y_{n-1,n} + Y_{nn}. \end{aligned} \tag{10}$$

Since, by earlier arguments, $c_i^g = \sqrt{\mu\epsilon}\, Y_{ii}^g$, and $c_{ij}^m - \sqrt{\mu\epsilon}\, Y_{ij}$, we may substitute (9) in (10), and (10) in (11), and obtain the general design equations,

$$\begin{aligned} c_1^g/\epsilon &= \sqrt{\mu/\epsilon}\,(1-N)/L \\ c_{12}^m/\epsilon &= \sqrt{\mu/\epsilon}\,N/L \\ c_2/\epsilon &= \sqrt{\mu/\epsilon}\,C_2 \\ c_{23}^m/\epsilon &= \sqrt{\mu/\epsilon}\,Y_2 \\ c_3^g/\epsilon &= \sqrt{\mu/\epsilon}\,C_3 \\ &\vdots \\ c_i^g/\epsilon &= \sqrt{\mu/\epsilon}\,C_i \\ c_{i,i+1}^m/\epsilon &= \sqrt{\mu/\epsilon}\,Y_i \\ &\vdots \\ c_n^g/\epsilon &= \sqrt{\mu/\epsilon}\,C_n. \end{aligned}$$

These then are the exact general design equations with which one may design an interditial structure to have the same behavior, except for phase, as the transmission-line cascade of Fig. 3. It will be found that the design equations of Matthaei can be obtained from them by substituting on the left the approximate values of L, N, C_i, and Y_i which he obtained from a lumped constant lowpass prototype. Of course, if the interdigital filter is terminated in short-circuits at both ends, then $Y_{11} = C_1 + Y_1$, etc., and if both ends are open-circuited, $Y_{nn} = 1/L_n$, etc.

Comments

This analysis has been carried out on the assumption that alternate terminals of the interdigital structure were short- and open-circuited. If we assume that all of the "b" terminals are short-circuited while all of the "a" terminals, except those at the input and output, are open-circuited, then (1) reduces to the node equations of a ladder network consisting only of capacitors. On the other hand, if all of the "b" terminals are open-circuited, while the internal "a" terminals are open-circuited, (1) reduces to the node equations of a ladder network consisting only of inductors. Of course, many networks in which the conductors are terminated in more general admittances may be analyzed in the same way.

The node equations of ladder networks containing only capacitors or only inductors have the same form as (8) except that all terms have a common frequency behavior. Following Guillemin, one may multiply the same row and column of these equations by a constant without altering the overall behavior of the network and thereby obtain equivalent networks. One may extend this equivalence operation to the node equations of transmission-line cascades thereby constructing a whole class of equivalent networks. In fact, transmission-line cascades consisting only of shunt capacitors separated by transmission-line elements are in a one-to-one correspondence[5] to ladder networks consisting only of capacitors, with the property that the correspondence of networks is uneffected by equal row and column multiplication by a constant. It follows that the Kuroda identity

[4] It should be noted that the capacity to ground of the ith conductor is defined with all of the conductors at the same potential while the mutual capacities are defined with all but one of the conductors at ground potential.

[5] This isomorphism underlies Wenzel's treatment of transmission-line cascades as capacitance networks.

[8] relating to shunt capacitors and transmission-line elements applies as well to pi sections consisting only of capacitors. Of course, the same remarks are valid for the dual situation involving series inductors.

Appendix

The impedance equations for the array of parallel conductors of Fig. 1, subject to the requirement that there be no coupling between nonadjacent conductors, may be written in the general case as,

$$\begin{vmatrix} V_{1a} \\ V_{1b} \\ V_{2a} \\ V_{2b} \\ \vdots \\ V_{na} \\ V_{nb} \end{vmatrix} = p \begin{vmatrix} Z_1 & Z_1 t & Z_1K_{12} & Z_1K_{12}t & \cdots & Z_1K_{1n} & Z_1K_{1n}t \\ Z_1 t & Z_1 & Z_1K_{12}t & Z_1K_{12} & \cdots & Z_1K_{1n}t & Z_1K_{1n} \\ Z_2K_{21} & Z_2K_{21}t & Z_2 & Z_2 t & \cdots & Z_2K_{2n} & Z_2K_{2n}t \\ Z_2K_{21}t & Z_2K_{21} & Z_2 t & Z_2 & \cdots & Z_2K_{2n}t & Z_2K_{2n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ Z_nK_{n1} & Z_nK_{n1}t & \cdot & \cdot & \cdots & Z_n & Z_n t \\ Z_nK_{n1}t & Z_nK_{n1} & \cdot & \cdot & \cdots & Z_n t & Z_n \end{vmatrix} \cdot \begin{vmatrix} I_{1a} \\ I_{1b} \\ I_{2a} \\ I_{2b} \\ \vdots \\ I_{na} \\ I_{nb} \end{vmatrix} \qquad (11)$$

for suitable definition of the symbols involved. Now V_{ia}, V_{ib}, I_{1b}, p and t have the same meanings as were used earlier. Z_i is the input impedance of the ith conductor when the other input terminals are open circuited and all the conductors are infinite in length. Also, $K_{ij}=K_i \cdots K_{j-1}\sqrt{Z_j/Z_i}$ and $K_{ji}=K_i \cdots K_{j-1}\sqrt{Z_i/Z_j}$ for $i<j+1$ while $K_{i,i+1}=K_i\sqrt{Z_j/Z_i}$ and $K_{i+1,i}=K_i\sqrt{Z_i/Z_j}$. Here K_iZ_{i+1}/Z_i is the voltage coupling factor, as defined by Bolljahn and Matthaei, which gives the ratio between the voltage at the $i+1$st terminal and the ith terminal. It will be seen that $K_{ik}=K_{ij}\cdot K_{jk}$ for $i<j<k$. Thus, Condition (4) of Wenzel is satisfied and there is no direct coupling between nonadjacent conductors. Moreover, $Z_jK_{ji}=Z_iK_{ij}$ so that the impedance matrix in (11) is symmetrical. For $Z_i=Z_j$, $i\neq j$, (11) reduces immediately to (1) of Wenzel subject to the assumption concerning coupling between nonadjacent conductors.

The admittance equations (1) will be a general representation of the array of parallel conductor between ground planes of Fig. 1 which is consistent with the impedance equation representation (11) if it can be shown that their admittance and impedance matrices are reciprocals. This can be done without great difficulty, if we assume that coefficients of the admittance equations are given in term of the coefficients in the impedance equations as follows:

$$Y_{11} = (1 - K_1^2)^{-1}Z_1^{-1}$$

$$Y_{ii} = (1 - K_{i-1}^2K_i^2)(1 - K_{i-1}^2)^{-1}(1 - K_i^2)^{-1}Z_i^{-1}, \quad i \neq 1 \text{ or } n$$

$$Y_{nn} = (1 - K_{n-1}^2)^{-1}Z_n^{-1} \qquad (12)$$

and

$$Y_{i,i+1} = -K_i(1 - K_i^2)^{-1}(Z_iZ_{i+1})^{-1/2}.$$

The evaluation of the terms of the product of the admittance matrix of (1) and the impedance matrix of (11) is simplified by the fact that each of the terms in the product is the sum of at most six nonzero components. This is apparent from the form of (1). Now, the term in the $2i-1$st row and $2i-1$st column of the product has the value,

$$p^2\left(\frac{K_{i-1}^2}{1 - K_{i-1}^2} - \frac{1 - K_{i-1}^2K_i^2}{(1 - K_{i-1}^2)(1 - K_i^2)} + \frac{K_i^2}{1 - K_i^2}\right)\cdot(t^2 - 1) = 1.$$

The term in the $2i+1$st row and $2i-1$st column contains the factor,

$$\frac{-K_{i-1}^2K_i}{1 - K_{i-1}^2} + \frac{K_i(1 - K_{i-1}^2K_i^2)}{(1 - K_{i-1}^2)(1 - K_i^2)} - \frac{K_i}{1 - K_i^2},$$

and vanishes because this factor vanishes. No other terms in the $2i-1$st column require evaluation since they can be seen to be zero by inspection. The terms in the $2i$th column of the product may be evaluated in the same way with similar results. The terms in the first two and last two rows and columns of the product matrix involve only four components and so require special consideration; but, now the evaluation is simpler and the final result is the same. Thus, the only nonzero elements in the product of the impedance and admittance matrices are the ones on the principal diagonal.

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